SCALED ASYMPTOTICS FOR SOME q-SERIES AS q APPROACHES ONE

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Abstract. In this work we investigate Plancherel-Rotach type asymptotics for some q-series as $q \to 1$. These q-series generalize Ramanujan function $A_q(z)$ (q-Airy function), Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, Ismail-Masson orthogonal polynomials(q^{-1}-Hermite polynomials) $h_n(x|q)$, Stieltjes-Wigert orthogonal polynomials $S_n(x;q)$, q-Laguerre orthogonal polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series.

1. Introduction

In [8] we derived certain Plancherel-Rotach type asymptotics for some q-series. These q-series generalize Ramanujan's entire function $A_q(z)$, Jackson's q-Bessel function $J_\nu^{(2)}(z;q)$, Ismail-Masson orthogonal polynomials (q^{-1}-Hermite polynomials) $h_n(x|q)$, Stieltjes-Wigert orthogonal polynomials $S_n(x;q)$, q-Laguerre orthogonal polynomials $L_n^{(\alpha)}(x;q)$ and confluent basic hypergeometric series.

In this work we employ the method used in [9] to study the scaled asymptotics of these q-series as $q \to 1$. In section §2 we list some notations. We present our results in section §3 and their proofs in section §4. Throughout this work we always assume that $0 < q < 1$ unless otherwise stated.

2. Preliminaries

For a complex number z, we define [1, 2, 4, 5]

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k),$$

and the q-Gamma function is defined as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z} \quad z \in \mathbb{C}.$$

The q-shifted factorials of $a, a_1, \dots a_m$ are given by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad (a_1, \dots, a_m;q)_n = \prod_{k=1}^{m}(a_k;q)_n$$

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for all integers $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. Given two sets of complex numbers $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$, the basic hypergeometric series ${}_r\phi_s$ is formally defined as

$$ {}_r\phi_s \left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} |q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k (zq^{-\ell})^k q^{\ell k^2}}{(q, b_1, \ldots, b_s; q)_k (-1)^{k(s+1-r)}}, $$

where

$$ \ell = \frac{s+1-r}{2}, $$

and it is a confluent basic hypergeometric series if $\ell > 0$.

Given nonnegative integers r, s, t and a positive number ℓ, we define [8]

$$ \begin{aligned} & g(a_1, \ldots, a_r; b_1, \ldots, b_s; q; \ell; z) \\ & = \sum_{k=0}^{\infty} \frac{(q^{k+1}, b_1 q^k, \ldots, b_s q^k; q)_\infty q^{\ell k^2} (-z)^k}{(a_1 q^k, \ldots, a_r q^k; q)_\infty}, \\ & h(a_1, \ldots, a_r; b_1, \ldots, b_s; c_1, \ldots, c_t; q; \ell; z) \\ & = \sum_{k=0}^{n} \frac{(q^{k+1}, b_1 q^k, \ldots, b_s q^k; q)_\infty q^{\ell k^2} (-z)^k}{(a_1 q^k, \ldots, a_r q^k; q)_\infty} \frac{(q, c_1, \ldots, c_t; q)_n}{(q, c_1, \ldots, c_t; q)_{n-k}}, \end{aligned} $$

where

$$ 0 \le a_1, \ldots, a_r, b_1, \ldots, b_s, c_1, \ldots, c_t < 1. \tag{2.1} $$

Jackson's q-Bessel function $J_\nu^{(2)}(z; q)$ is defined as [4, 2, 5]

$$ J_\nu^{(2)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k\nu} (-1)^k}{(q, q^{\nu+1}; q)_k} \left(\frac{z}{2} \right)^{2k+\nu}, \quad \nu > -1. $$

The Ismail-Masson polynomials $\{h_n(x|q)\}_{n=0}^{\infty}$ are defined as [4]

$$ h_n(\sinh \xi | q) = \sum_{k=0}^{n} \frac{(q; q)_n q^{k(k-n)} (-1)^k e^{(n-2k)\xi}}{(q; q)_k (q; q)_{n-k}}. $$

Stieltjes-Wigert orthogonal polynomials $\{S_n(x; q)\}_{n=0}^{\infty}$ are defined as [4]

$$ S_n(x; q) = \sum_{k=0}^{n} \frac{q^{k^2} (-x)^k}{(q; q)_k (q; q)_{n-k}}. $$

The q-Laguerre orthogonal polynomials $\left\{ L_n^{(\alpha)}(x; q) \right\}_{n=0}^{\infty}$ are defined as [4]

$$ L_n^{(\alpha)}(x; q) = \sum_{k=0}^{n} \frac{q^{k^2+\alpha k} (-x)^k (q^{\alpha+1}; q)_n}{(q; q)_k (q, q^{\alpha+1}; q)_{n-k}} $$

for $\alpha > -1$. Clearly, we have

$$\begin{aligned}
A_q(z) &= \frac{g(-;-;q;1;z)}{(q;q)_\infty}, \\
J_\nu^{(2)}(z;q) &= \frac{g(-;q^{\nu+1};q;1;z^2q^\nu/4)}{(q;q)_\infty^2(2/z)^\nu}, \\
h_n(\sinh\xi|q) &= \frac{h(-;-;-;q;1;e^{-2\xi}q^{-n})}{e^{-n\xi}(q;q)_\infty}, \\
S_n(x;q) &= \frac{h(-;-;-;q;1;x)}{(q;q)_n(q;q)_\infty}, \\
L_n^{(\alpha)}(x;q) &= \frac{h(-;-;q^{\alpha+1};q;1;xq^\alpha)}{(q;q)_n(q;q)_\infty}, \\
{}_r\phi_s\left(\begin{array}{c} a_1,\dots,a_r \\ b_1,\dots,b_s \end{array} |q, z(-1)^{s-r}\right) &= \frac{(a_1,\dots,a_r;q)_\infty g(a_1,\dots,a_r;b_1,\dots b_s;q;\ell;zq^{-\ell})}{(q,b_1,\dots,b_s;q)_\infty}.
\end{aligned}$$

The four Jacobi theta functions are [7]

$$\begin{aligned}
\theta_1(v|\tau) &= -i\sum_{k=-\infty}^{\infty}(-1)^k q^{(k+1/2)^2}e^{(2k+1)\pi iv}, \\
\theta_2(v|\tau) &= \sum_{k=-\infty}^{\infty} q^{(k+1/2)^2}e^{(2k+1)\pi iv}, \\
\theta_3(v|\tau) &= \sum_{k=-\infty}^{\infty} q^{k^2}e^{2k\pi iv}, \\
\theta_4(v|\tau) &= \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2}e^{2k\pi iv},
\end{aligned}$$

where

$$q = e^{\pi i\tau}, \quad \Im(\tau) > 0.$$

For our convenience, we also use the following notations

$$\theta_\lambda(z;q) = \theta_\lambda(v|\tau), \quad z = e^{2\pi iv}, \quad q = e^{\pi i\tau}$$

with

$$\lambda = 1,2,3,4.$$

By the Jacobi's triple product formula it follows that

$$\begin{aligned}
\theta_1(v|\tau) &= 2q^{1/4}\sin\pi v(q^2;q^2)_\infty(q^2e^{2\pi iv};q^2)_\infty(q^2e^{-2\pi iv};q^2)_\infty, \\
\theta_2(v|\tau) &= 2q^{1/4}\cos\pi v(q^2;q^2)_\infty(-q^2e^{2\pi iv};q^2)_\infty(-q^2e^{-2\pi iv};q^2)_\infty, \\
\theta_3(v|\tau) &= (q^2;q^2)_\infty(-qe^{2\pi iv};q^2)_\infty(-qe^{-2\pi iv};q^2)_\infty, \\
\theta_4(v|\tau) &= (q^2;q^2)_\infty(qe^{2\pi iv};q^2)_\infty(qe^{-2\pi iv};q^2)_\infty.
\end{aligned}$$

The Jacobi θ functions satisfy transformations

$$\theta_1\left(\frac{v}{\tau}\mid -\frac{1}{\tau}\right) = -i\sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_1\left(v\mid\tau\right),$$
$$\theta_2\left(\frac{v}{\tau}\mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_4\left(v\mid\tau\right),$$
$$\theta_3\left(\frac{v}{\tau}\mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_3\left(v\mid\tau\right),$$
$$\theta_4\left(\frac{v}{\tau}\mid -\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\pi i v^2/\tau}\theta_2\left(v\mid\tau\right).$$

The Euler Gamma function $\Gamma(z)$ is given by [1, 2, 4, 5]

$$\frac{1}{\Gamma(z)} = z\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{k}\right)^{-z}, \quad z\in\mathbb{C}.$$

For any real number x, we have

$$x = \lfloor x\rfloor + \{x\},$$

where the fractional part of x is $\{x\}\in[0,1)$ and $\lfloor x\rfloor\in\mathbb{Z}$ is the greatest integer less or equal to x. The arithmetic function

$$\chi(n) = \begin{cases} 1 & 2\nmid n \\ 0 & 2\mid n \end{cases},$$

which is the principal character modulo 2, satisfies the identities

$$\chi(n) = 2\left\{\frac{n}{2}\right\} = n - 2\left\lfloor\frac{n}{2}\right\rfloor = \left\lfloor\frac{n+1}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor.$$

Thus,

$$\left\lfloor\frac{n+1}{2}\right\rfloor = \frac{n+\chi(n)}{2},$$

and

$$\left\lfloor\frac{n}{2}\right\rfloor = \frac{n-\chi(n)}{2}.$$

3. Main Results

Definition 1. An admissible scale is a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive numbers such that

$$\lim_{n\to\infty}\frac{\lambda_n}{\log n} = \infty, \quad \lim_{n\to\infty}\frac{n}{\lambda_n^2} = \infty. \tag{3.1}$$

Clearly,

$$\lambda_n = n^{\beta}\log^{\gamma} n, \quad 0<\beta<\frac{1}{2}, \quad \gamma\ge 0,$$

and

$$\lambda_n = \log^{\gamma} n, \quad \gamma>1$$

are admissible scales.

3.1. **g-function.** To simplify the type setting in the following theorem, we let

$$g(z;q)=g(a_1,\dots,a_r;b_1,\dots,b_s;q;\ell;z).$$

Theorem 2. *Given an admissible scale λ_n, assume that*

$$z=e^{2\pi v},\quad q=e^{-\pi\lambda_n^{-1}},\quad \ell>0,\quad v\in\mathbb{R},$$

and

$$a_j=q^{\alpha_j},\quad b_k=q^{\beta_k},\quad \alpha_j,\beta_k>0$$

for

$$1\le j\le r,\quad 1\le k\le s.$$

Then,

$$\begin{aligned}g(-q^{-4n\ell}z;q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2\right\}\\&\quad\times\sqrt{\frac{\lambda_n}{\ell}}\left\{1+\mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\},\end{aligned}$$

and

$$\begin{aligned}g(q^{-4n\ell}z;q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2-\frac{\pi\lambda_n}{4\ell}\right\}\\&\quad\times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n v}{\ell}+\mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\}\end{aligned}$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Ramanujan's entire function we have:

Corollary 3. *Given an admissible scale λ_n , assume that*

$$z=e^{2\pi v},\quad q=e^{-\pi\lambda_n^{-1}},\quad v\in\mathbb{R},$$

we have

$$\begin{aligned}A_q(-q^{-4n}z)&=\exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2+\frac{\pi\lambda_n}{6}-\frac{\pi}{24\lambda_n}\right\}\\&\quad\times\frac{1}{\sqrt{2}}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},\end{aligned}$$

and

$$\begin{aligned}A_q(q^{-4n}z)&=\exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2-\frac{\pi\lambda_n}{12}-\frac{\pi}{24\lambda_n}\right\}\\&\quad\times\sqrt{2}\left\{\cos\pi\lambda_n v+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}\end{aligned}$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the Jackson's q-Bessel function we have:

Corollary 4. *For an admissible scale λ_n, assume that*

$$z = e^{2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R}, \quad \nu > -1,$$

then,

$$\begin{aligned} J_\nu^{(2)}(2i\sqrt{zq^{-\nu}}q^{-2n}; q) &= \frac{\exp\left(\frac{\pi\lambda_n}{3} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n} + \frac{\nu\pi i}{2}\right)}{2\sqrt{\lambda_n}} \\ &\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\} \\ &\times \left\{1 + \mathcal{O}(e^{-\pi\lambda_n})\right\}, \end{aligned}$$

and

$$\begin{aligned} J_\nu^{(2)}(2\sqrt{zq^{-\nu}}q^{-2n}; q) &= \frac{\exp\left(\frac{\pi\lambda_n}{12} - \frac{\pi}{12\lambda_n} + \frac{\nu^2\pi}{4\lambda_n}\right)}{\sqrt{\lambda_n}} \\ &\times \exp\left\{\pi\lambda_n\left(v + \frac{4n+\nu}{2\lambda_n}\right)^2\right\} \\ &\times \left\{\cos\pi\lambda_n v + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the confluent basic hypergeometric series we have:

Corollary 5. *Given an admissible scale λ_n, assume that*

$$z = e^{2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R},$$

and

$$\alpha_j, \beta_k > 0, \quad 1 \le j \le r, \quad 1 \le k \le s.$$

Let

$$\ell = \frac{s+1-r}{2} > 0, \quad \rho = \sum_{j=1}^{r}\alpha_j - \sum_{k=1}^{s}\beta_k - 1,$$

then,

$$\begin{aligned} & {}_r\phi_s\left(\begin{matrix} q^{\alpha_1}, \dots, q^{\alpha_r} \\ q^{\beta_1}, \dots, q^{\beta_s} \end{matrix}\ |q, (-1)^{s+1-r}zq^{-\ell(4n-1)}\right) \\ &= \frac{\prod_{k=1}^{s}\Gamma(\beta_k)}{\prod_{j=1}^{r}\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell}2^\ell\pi^{\rho+2\ell}} \\ &\times \left\{\exp\frac{\pi\lambda_n}{\ell}\left(v + \frac{2n\ell}{\lambda_n}\right)^2 + \ell\pi\lambda_n/3\right\} \\ &\times \left\{1 + \mathcal{O}\left(\lambda_n^{-1}\right)\right\}, \end{aligned}$$

and

$$
\begin{aligned}
&{}_r\phi_s\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s}\end{matrix}\ |q,(-1)^{s-r}zq^{-\ell(4n-1)}\right) \\
&= \frac{\prod_{k=1}^{s}\Gamma(\beta_k)}{\prod_{j=1}^{r}\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell}2^{\ell-1}\pi^{\rho+2\ell}} \\
&\times\left\{\exp\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2+\frac{\ell\pi\lambda_n}{3}-\frac{\pi\lambda_n}{4\ell}\right\} \\
&\times\left\{\cos\frac{\pi\lambda_n v}{\ell}+\mathcal{O}\left(\lambda_n^{-1}\right)\right\}
\end{aligned}
$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

3.2. **h-function.** For our convenience we let

$$h_n(z;q)=h_\ell(a_1,\dots,a_r;b_1,\dots,b_s;c_1,\dots,c_t;q;z).$$

We have similar results for the h function:

Theorem 6. *Given an admissible scale λ_n , assume that*

$$z=e^{2\pi v},\quad q=e^{-\pi\lambda_n^{-1}},\quad \ell>0,\quad v\in\mathbb{R},$$

and

$$a_j=q^{\alpha_j},\quad b_k=q^{\beta_k},\quad \alpha_j,\beta_k>0$$

for

$$1\le j\le r,\quad 1\le k\le s.$$

Then,

$$
\begin{aligned}
h(-zq^{-n\ell};q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2+\frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}\right\} \\
&\times\sqrt{\frac{\lambda_n}{\ell}}\left\{1+\mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
h(zq^{-n\ell};q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2+\frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi\lambda_n}{4\ell}\right\} \\
&\times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)+\mathcal{O}(e^{-2\ell^{-1}\pi\lambda_n})\right\}
\end{aligned}
$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For Ismail-Masson orthogonal polynomials we have:

Corollary 7. *Given an admissible scale λ_n, for any $v\in\mathbb{R}$, we have*

$$
\begin{aligned}
h_n\left(\sinh\pi\left(v+\frac{i}{2}\right)\mid q\right)&=\frac{\exp\left\{\frac{\pi n^2}{4\lambda_n}+\frac{\pi\lambda_n}{6}-\frac{\pi(1+12\chi(n))}{24\lambda_n}\right\}}{(-i)^n\sqrt{2}} \\
&\times\left\{\exp\left[\pi\lambda_n\left(v-\frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\}\left\{1+\mathcal{O}(e^{-\pi\lambda_n})\right\},
\end{aligned}
$$

and

$$h_n(\sinh \pi v \mid q) = (-1)^n \sqrt{2} \exp\left\{\frac{n^2\pi}{4\lambda_n} - \frac{(1+12\chi(n))\pi}{24\lambda_n} - \frac{\pi\lambda_n}{12}\right\} \times \left\{\exp\left[\pi\lambda_n\left(v - \frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\} \times \left\{\cos \pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right) + \mathcal{O}(e^{-2\pi\lambda_n})\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For Stieltjes-Wigert orthogonal polynomials we have:

Corollary 8. *Given an admissible scale λ_n, assume that*

$$z = e^{2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R}.$$

Then,

$$S_n(-zq^{-n}; q) = \frac{\exp\left\{\frac{\pi\lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}} \times \left\{\exp \pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},$$

and

$$S_n(zq^{-n}; q) = \frac{\exp\left\{\frac{\pi\lambda_n}{12} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}} \times \left\{\exp \pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\} \times \left\{\cos \pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right) + \mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n \to \infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

For the q-Laguerre orthogonal polynomials we have:

Corollary 9. *Given an admissible scale λ_n, assume that*

$$z = e^{-2\pi v}, \quad q = e^{-\pi\lambda_n^{-1}}, \quad v \in \mathbb{R}, \quad \alpha > -1.$$

Then,

$$L_n^{(\alpha)}(-zq^{-\alpha-n}; q) = \frac{\exp\left\{\frac{\pi\lambda_n}{3} + \frac{\pi(n-1)\chi(n)}{2\lambda_n} - \frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}} \times \left\{\exp \pi\lambda_n\left(v + \frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1 + \mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},$$

and

$$L_n^{(\alpha)}(zq^{-\alpha-n};q)=\frac{\exp\left\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}}$$
$$\times\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}$$
$$\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}$$

as $n\to\infty$, and the $\mathcal{O}$-term is uniform for v in any compact subset of $\mathbb{R}$.

Remark 10. Similar results hold for general τ and β defined in [8] and their proofs are also similar to the proofs for the special cases here. However, we feel that the formulas for the special cases are more appealing and thus skip the general formulas.

4. Proofs

The following lemma is from [8]:

Lemma 11. *Given $a\in\mathbb{C}$ with*

$$0<\frac{|a|\,q^n}{1-q}<\frac{1}{2}$$

for some $n\in\mathbb{N}$. Then,

$$\frac{(a;q)_\infty}{(a;q)_n}=(aq^n;q)_\infty=1+r_1(a;n)$$

with

$$|r_1(a;n)|\le\frac{2\,|a|\,q^n}{1-q}$$

and

$$\frac{(a;q)_n}{(a;q)_\infty}=\frac{1}{(aq^n;q)_\infty}=1+r_2(a;n)$$

with

$$|r_2(a;n)|\le\frac{2\,|a|\,q^n}{1-q}.$$

We also need the following lemma:

Lemma 12. *Given a sequence of positive numbers $\{\lambda_n\}_{n=1}^\infty$ with $\lim_{n\to\infty}\lambda_n=\infty$. let*

$$q=e^{-\pi\lambda_n^{-1}},\quad x>0,$$

then,

$$(q^x;q)_\infty=\frac{\sqrt{2}\pi^{1-x}\lambda_n^{x-1/2}}{\Gamma(x)\exp(\pi\lambda_n/6)}\left\{1+\mathcal{O}\left(\lambda_n^{-1}\right)\right\}$$

as $n\to\infty$.

Proof. For $x \neq 0, -1, -2, \ldots$ and let $q = e^{-t}$, the McIntosh asymptotic formula [6] is

$$\begin{aligned}\log(q^x;q)_\infty &= -\frac{\pi^2}{6t} + \left(\frac{1}{2} - x\right)\log t + \frac{\log(2\pi)}{2} - \log\Gamma(x) \\ &\quad + \sum_{k=1}^{p} \frac{B_k B_{k+1}(x)}{k(k+1)!} t^k + \mathcal{O}\left(t^{p+1}\right)\end{aligned}$$

for any positive integer p as $t \to 0^+$, where B_k is the k^{th} Bernoulli number and $B_k(x)$ is the k^{th} Bernoulli polynomial. Take the main term in the McIntosh asymptotic formula with $t = \frac{\pi}{\lambda_n}$ and Lemma 12 follows. □

Take $\lambda = 0, \quad \tau = 2, \quad m = 2n$ in Theorem 2.2 of [9] we get the following result:

Lemma 13. *Assume that $z \in \mathbb{C}\setminus\{0\}$, $\ell > 0$ and (2.1), then,*

$$g(q^{-4n\ell}z;q) = z^{2n} q^{-4n^2\ell}\left\{\theta_4\left(z^{-1};q^\ell\right) + r_g(n|1)\right\},$$

and

$$|r_g(n|1)| \leq \frac{2^{s+r+3}\theta_3\left(|z|^{-1};q^\ell\right)}{(a_1,\ldots,a_r;q)_\infty}\left\{\frac{q^{n+1}}{1-q} + \frac{q^{\ell n^2}}{|z|^n}\right\}$$

for n sufficiently large. In particular,

$$\begin{aligned}&{}_r\phi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s\end{matrix}\ \Big|q, (-1)^{s-r}zq^{-4n\ell}\right) \\ &= \frac{(a_1,\ldots,a_r;q)_\infty z^{2n}\left\{\theta_4\left(z^{-1}q^\ell;q^\ell\right) + r_\phi(n|1)\right\}}{(q,b_1,\ldots,b_s;q)_\infty q^{2\ell n(2n+1)}},\end{aligned}$$

and

$$|r_\phi(n|1)| \leq \frac{2^{s+r+3}\theta_3\left(|z|^{-1}q^\ell;q^\ell\right)}{(a_1,\ldots,a_r;q)_\infty}\left\{\frac{q^{n+1}}{1-q} + \frac{q^{\ell n^2+\ell n}}{|z|^n}\right\}$$

for n sufficiently large, where

$$\ell = \frac{s+1-r}{2} > 0.$$

Similarly, if we take $\lambda = 0$ and $\tau = \frac{1}{2}$ in Theorem 2.4 of [9] we get

Lemma 14. *Assume that $z \in \mathbb{C}\setminus\{0\}$, $\ell > 0$ and (2.1), then*

$$h_n(zq^{-n\ell};q) = (-z)^{\lfloor n/2\rfloor} q^{-\ell\left[n^2-\chi(n)\right]/4}\left\{\theta_4\left(z^{-1};q^\ell\right) + r_h(n|1)\right\},$$

and

$$\begin{aligned}|r_h(n|1)| &\leq \frac{2^{s+r+2t+5}\theta_3(|z|^{-1};q^\ell)}{(a_1,\ldots,a_r;q)_\infty} \\ &\quad \times\left\{\frac{q^{\lfloor n/4\rfloor+1}}{1-q} + |z|^{\lfloor n/4\rfloor} q^{\ell\lfloor n/4\rfloor^2} + \frac{q^{\ell\lfloor n/4\rfloor^2}}{|z|^{\lfloor n/4\rfloor}}\right\}\end{aligned}$$

for n sufficiently large.

4.1. **Proof for Theorem 2.** From the formulas of θ_3 to obtain

$$\begin{aligned}\theta_3(e^{-2\pi v};e^{-\pi\ell\lambda_n^{-1}}) &= \theta_3\left(vi|\ell\lambda_n^{-1}i\right) \\ &= \sqrt{\frac{\lambda_n}{\ell}}e^{\pi\ell^{-1}\lambda_n v^2}\theta_3\left(\frac{\lambda_n v}{\ell}\mid\frac{\lambda_n i}{\ell}\right) \\ &= \sqrt{\frac{\lambda_n}{\ell}}e^{\pi\ell^{-1}\lambda_n v^2}\left\{1+\mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\}\end{aligned}\tag{4.1}$$

as $n\to\infty$, uniformly for all $v\in\mathbb{R}$. Clearly we have

$$\frac{q^{n+1}}{1-q}+q^{\ell n^2}e^{-2n\pi v}=\mathcal{O}(\lambda_n e^{-\pi n\lambda_n^{-1}})$$

as $n\to\infty$, uniformly for v in any compact subset of $\mathbb{R}$. From Lemma 12 we have

$$(q^{\alpha_1},\ldots,q^{\alpha_r};q)_\infty=\frac{2^{r/2}\pi^{r-\sum_{j=1}^r\alpha_j}\left\{1+\mathcal{O}(\lambda_n^{-1})\right\}}{e^{r\pi\lambda_n/6}\lambda_n^{r/2-\sum_{j=1}^r\alpha_j}\prod_{j=1}^r\Gamma(\alpha_j)}\tag{4.2}$$

as $n\to\infty$. Condition (3.1) gives

$$g(-q^{-4n\ell}z;q)=\sqrt{\frac{\lambda_n}{\ell}}\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2\right\}\left\{1+\mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\}$$

as $n\to\infty$, uniformly for v in any compact subset of $\mathbb{R}$.

Since

$$\begin{aligned}\theta_4\left(z^{-1};q^\ell\right) &= \theta_4\left(e^{-2\pi v};e^{-\ell\pi\lambda_n^{-1}}\right) \\ &= \theta_4\left(vi\mid\frac{\ell i}{\lambda_n}\right) \\ &= \sqrt{\frac{\lambda_n}{\ell}}e^{\pi\ell^{-1}\lambda_n v^2}\theta_2\left(\frac{\lambda_n v}{\ell}\mid\frac{i\lambda_n}{\ell}\right) \\ &= 2\sqrt{\frac{\lambda_n}{\ell}}\exp\left(\frac{\pi\lambda_n v^2}{\ell}-\frac{\pi\lambda_n}{4\ell}\right) \\ &\quad\times\cos\frac{\pi\lambda_n v}{\ell}\left\{1+\mathcal{O}\left(e^{-2\pi\ell^{-1}\lambda_n}\right)\right\},\end{aligned}\tag{4.3}$$

as $n\to\infty$, uniformly in $v\in\mathbb{R}$. Thus,

$$\begin{aligned}g(q^{-4n\ell}z;q) &= \exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2-\frac{\pi\lambda_n}{4\ell}\right\} \\ &\quad\times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n v}{\ell}+\mathcal{O}\left(e^{-2\ell^{-1}\pi\lambda_n}\right)\right\}\end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.2. **Proof for Corollary 3.** By Lemma 12 and Theorem 2 we have

$$\begin{aligned} A_q(-q^{-4n}z) &= \frac{g(-;-;q;1;-q^{-4n}z)}{(q;q)_\infty} \\ &= \exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2+\frac{\pi\lambda_n}{6}-\frac{\pi}{24\lambda_n}\right\} \\ &\times \frac{1}{\sqrt{2}}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned} A_q(q^{-4n}z) &= \frac{g(-;-;q;1;q^{-4n}z)}{(q;q)_\infty} \\ &= \exp\left\{\pi\lambda_n\left(v+\frac{2n}{\lambda_n}\right)^2-\frac{\pi\lambda_n}{12}-\frac{\pi}{24\lambda_n}\right\} \\ &\times \sqrt{2}\left\{\cos\pi\lambda_n v+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.3. **Proof for Corollary 4.** Apply Lemma 12 and Theorem 2 to get

$$\begin{aligned} J_\nu^{(2)}(2i\sqrt{zq^{-\nu}}q^{-2n};q) &= \frac{g(-;q^{\nu+1};q;1;-zq^{-4n})}{(q;q)_\infty^2(i\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}} \\ &= \frac{\exp\left(\frac{\pi\lambda_n}{3}-\frac{\pi}{12\lambda_n}+\frac{\nu^2\pi}{4\lambda_n}+\frac{\nu\pi i}{2}\right)}{2\sqrt{\lambda_n}} \\ &\times \exp\left\{\pi\lambda_n\left(v+\frac{4n+\nu}{2\lambda_n}\right)^2\right\} \\ &\times \left\{1+\mathcal{O}(e^{-\pi\lambda_n})\right\}, \end{aligned}$$

and

$$\begin{aligned} J_\nu^{(2)}(2\sqrt{zq^{-\nu}}q^{-2n};q) &= \frac{g(-;q^{\nu+1};q;1;zq^{-4n})}{(q;q)_\infty^2(\sqrt{zq^{-\nu}}q^{-2n})^{-\nu}} \\ &= \frac{\exp\left(\frac{\pi\lambda_n}{12}-\frac{\pi}{12\lambda_n}+\frac{\nu^2\pi}{4\lambda_n}\right)}{\sqrt{\lambda_n}} \\ &\times \exp\left\{\pi\lambda_n\left(v+\frac{4n+\nu}{2\lambda_n}\right)^2\right\} \\ &\times \left\{\cos\pi\lambda_n v+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\} \end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.4. **Proof for Corollary 5.** Apply Lemma 12 , Lemma 12 and Theorem 2 to get

$$
\begin{aligned}
&{}_r\phi_s\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s}\end{matrix}\ |q,(-1)^{s+1-r}zq^{-\ell(4n-1)}\right)\\
&=\frac{(q^{\alpha_1},\dots,q^{\alpha_r};q)_\infty g(-q^{-4n\ell}z;q)}{(q,q^{\beta_1},\dots,q^{\beta_s};q)_\infty}\\
&=\frac{\prod_{k=1}^s\Gamma(\beta_k)}{\prod_{j=1}^r\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{2^\ell\pi^{\rho+2\ell}\sqrt{\ell}}\\
&\times\left\{\exp\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2+\ell\pi\lambda_n/3\right\}\\
&\times\left\{1+\mathcal{O}\left(\lambda_n^{-1}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_s\phi_r\left(\begin{matrix} q^{\alpha_1},\dots,q^{\alpha_r} \\ q^{\beta_1},\dots,q^{\beta_s}\end{matrix}\ |q,(-1)^{s-r}zq^{-\ell(4n-1)}\right)\\
&=\frac{(q^{\alpha_1},\dots,q^{\alpha_r};q)_\infty g(q^{-4n\ell}z;q)}{(q,q^{\beta_1},\dots,q^{\beta_s};q)_\infty}\\
&=\frac{\prod_{k=1}^s\Gamma(\beta_k)}{\prod_{j=1}^r\Gamma(\alpha_j)}\frac{\lambda_n^{\rho+\ell+1/2}}{\sqrt{\ell}2^{\ell-1}\pi^{\rho+2\ell}}\\
&\times\left\{\exp\frac{\pi\lambda_n}{\ell}\left(v+\frac{2n\ell}{\lambda_n}\right)^2+\frac{\ell\pi\lambda_n}{3}-\frac{\pi\lambda_n}{4\ell}\right\}\\
&\times\left\{\cos\frac{\pi\lambda_n v}{\ell}+\mathcal{O}\left(\lambda_n^{-1}\right)\right\}.
\end{aligned}
$$

4.5. **Proof for Theorem 6.** Clearly,

$$
\frac{q^{\lfloor n/4\rfloor+1}}{1-q}+|z|^{\lfloor n/4\rfloor}q^{\ell\lfloor n/4\rfloor^2}+\frac{q^{\ell\lfloor n/4\rfloor^2}}{|z|^{\lfloor n/4\rfloor}}=\mathcal{O}\left(e^{-\pi n/(4\lambda_n)}\right) \tag{4.4}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$. From equations (4.1), (4.2) and (4.4) to get

$$
\begin{aligned}
h(-zq^{-n\ell};q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2+\frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}\right\}\\
&\times\sqrt{\frac{\lambda_n}{\ell}}\left\{1+\mathcal{O}(e^{-\ell^{-1}\pi\lambda_n})\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$. Using equations (4.1), (4.2), (4.3) and (4.4) to obtain

$$
\begin{aligned}
h(zq^{-n\ell};q)&=\exp\left\{\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)^2+\frac{\ell\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi\lambda_n}{4\ell}\right\}\\
&\times 2\sqrt{\frac{\lambda_n}{\ell}}\left\{\cos\frac{\pi\lambda_n}{\ell}\left(v+\frac{\ell(n-\chi(n))}{2\lambda_n}\right)+\mathcal{O}(e^{-2\ell^{-1}\pi\lambda_n})\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.6. **Proof for Corollary 7.** For $v \in \mathbb{R}$, Lemma 12 and Theorem 6 implies

$$\begin{aligned} h_n\left(\sinh \pi\left(v+\frac{i}{2}\right) \mid q\right) &= \frac{h(-;-;-;q;1;-e^{2\pi v}q^{-n})}{(-i)^n e^{n\pi v}(q;q)_\infty} \\ &= \frac{\exp\left\{\frac{\pi n^2}{4\lambda_n}+\frac{\pi\lambda_n}{6}-\frac{\pi(1+12\chi(n))}{24\lambda_n}\right\}}{(-i)^n\sqrt{2}} \\ &\times\left\{\exp\left[\pi\lambda_n\left(v-\frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\}\left\{1+\mathcal{O}(e^{-\pi\lambda_n})\right\}, \end{aligned}$$

and

$$\begin{aligned} h_n(\sinh \pi v \mid q) &= \frac{h(-;-;-;q;1;e^{2\pi v}q^{-n})}{(-1)^n e^{n\pi v}(q;q)_\infty} \\ &= (-1)^n\sqrt{2}\exp\left\{\frac{n^2\pi}{4\lambda_n}-\frac{(1+12\chi(n))\pi}{24\lambda_n}-\frac{\pi\lambda_n}{12}\right\} \\ &\times\left\{\exp\left[\pi\lambda_n\left(v-\frac{\chi(n)}{2\lambda_n}\right)^2\right]\right\} \\ &\times\left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}(e^{-2\pi\lambda_n})\right\} \end{aligned}$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.7. **Proof for Corollary 8.** From Lemma11 and Lemma12 to obtain

$$\begin{aligned} \frac{1}{(q;q)_n(q;q)_\infty} &= \frac{1}{(q;q)_\infty^2}\frac{(q;q)_\infty}{(q;q)_n} \\ &= \frac{\exp\left\{\frac{\pi\lambda_n}{3}-\frac{\pi}{12\lambda_n}\right\}}{2\lambda_n}\left\{1+\mathcal{O}\left(e^{-4\pi\lambda_n}\right)\right\} \end{aligned} \tag{4.5}$$

as $n\to\infty$. Hence, Theorem 6 implies

$$\begin{aligned} S_n(-zq^{-n};q) &= \frac{h(-;-;-;q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty} \\ &= \frac{\exp\left\{\frac{\pi\lambda_n}{3}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}} \\ &\times\left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\}, \end{aligned}$$

and

$$
\begin{aligned}
S_n(zq^{-n};q) &= \frac{h(-;-;-;q;1;zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\left\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}} \\
&\times \left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\} \\
&\times \left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

4.8. **Proof for Corollary 9.** From (4.5) and Theorem 6 to obtain

$$
\begin{aligned}
L_n^{(\alpha)}(-zq^{-\alpha-n};q) &= \frac{h(-;-;q^{\alpha+1};q;1;-zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\left\{\frac{\pi\lambda_n}{3}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\right\}}{2\sqrt{\lambda_n}} \\
&\times \left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\}\left\{1+\mathcal{O}\left(e^{-\pi\lambda_n}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
L_n^{(\alpha)}(zq^{-\alpha-n};q) &= \frac{h(-;-;q^{\alpha+1};q;1;zq^{-n})}{(q;q)_n(q;q)_\infty} \\
&= \frac{\exp\left\{\frac{\pi\lambda_n}{12}+\frac{\pi(n-1)\chi(n)}{2\lambda_n}-\frac{\pi}{12\lambda_n}\right\}}{\sqrt{\lambda_n}} \\
&\times \left\{\exp\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)^2\right\} \\
&\times \left\{\cos\pi\lambda_n\left(v+\frac{n-\chi(n)}{2\lambda_n}\right)+\mathcal{O}\left(e^{-2\pi\lambda_n}\right)\right\}
\end{aligned}
$$

as $n\to\infty$, uniformly on any compact subset of $\mathbb{R}$.

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